Symplectic invariants of some families of Lagrangian T^3-fibrations

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Abstract

We construct families of Lagrangian 3-torus fibrations resembling the topology of some of the singularities in *Topological Mirror Symmetry* [8]. We perform a detailed analysis of the affine structure on the base of these fibrations near their discriminant loci. This permits us to classify the aforementioned families up to fibre preserving symplectomorphism. The kind of degenerations we investigate give rise to a large number of symplectic invariants.

1 Introduction

There is increasing interest in the geometry of Calabi-Yau manifolds. Much of this interest is motivated by an intriguing relation between pairs of Calabi-Yau manifolds called Mirror Symmetry. This relation interchanges– in a highly non-trivial way –the complex structure of a Calabi-Yau manifold, Y, with the symplectic structure of its mirror, $\check{Y}$. There are several approaches to Mirror Symmetry; one of these is proposed by Strominger, Yau and Zaslow (SYZ) [21]. The SYZ Conjecture claims– based on string theoretic arguments –that the mirror relation can be explained in terms of certain duality between T^n-fibrations on a pair of mirror Calabi-Yau manifolds.

The purpose of this paper is primarily motivated by the SYZ Conjecture; we are interested in the symplectic geometry of Calabi-Yau manifolds fibred by tori. Let (Y, J, ω) be a compact Kähler manifold of $\dim_{\mathbb{C}}(Y) = n$ with complex structure J and Kähler (symplectic) form ω. We say that Y is a Calabi-Yau manifold if the canonical line bundle has a non-vanishing global section Ω such that $c\Omega \wedge \bar{\Omega} = \omega^n$ for some constant c [9]. A very popular example of a Calabi-Yau 3-fold is the (smooth) quintic hypersurface in $\mathbb{P}^4$ defined by:

$$x_0x_1x_2x_3x_4 + t(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5) = 0, \tag{1}$$

where $t \in D_0 \subset \mathbb{C}$ a small punctured disk around 0.

A submanifold L of Y is called Lagrangian if $\omega|_L = 0$ and $\dim_{\mathbb{R}} L = n$. A Lagrangian submanifold L satisfying $\operatorname{Im}\Omega|_L = 0$, is called special Lagrangian. This term was coined by Harvey and Lawson [13].

A first attempt to state the SYZ Conjecture using mathematical language can be outlined as follows (c.f. [10], [5] and [6]):

Conjecture 1.1. Let Y and $\check{Y}$ be a mirror pair of Calabi-Yau n-folds satisfying certain additional conditions. Let B be a compact connected manifold. Then there exists C^∞ maps $f : Y \to B$ and $\check{f} : \check{Y} \to B$ such that for $b \in B$ the fibres $f^{-1}(b)$ and $\check{f}^{-1}(b)$ are special Lagrangian. There is a codimension 2 closed subset $\Delta \subseteq B$ such that the fibres $f^{-1}(b)$ and $\check{f}^{-1}(b)$ over $b \in B \setminus \Delta$ are dual n-tori.

The idea of duality in Conjecture 1.1 can be explained in the following way. First consider the T^n-bundle $f_0 : Y_0 \to B_0 = B \setminus \Delta$, resulting from removing the singular fibres of f. Let G be a group ($G = \mathbb{R}$ or $\mathbb{Z}$ for our purposes) and denote by $R^k f_{0*}(G)$ the locally constant sheaf on B_0 induced by the presheaf $\mathcal{R}^k f_{0*}(G) = \{U \mapsto H^k(f^{-1}(U), G), U \subseteq B\}$. Denote by $\mathcal{E} = R^1 f_{0*}(\mathbb{R}) \otimes C^\infty(B_0)$. This gives a rank n vector bundle $\mathcal{E} \to B_0$ with $R^1 f_{0*}(\mathbb{Z})$ being a family of rank n lattices lying inside $\mathcal{E}$. Letting $\check{Y}_0 = \mathcal{E}/R^1 f_{0*}(\mathbb{Z})$ we can define the dual of f_0 as the T^n-bundle:

$$\check{f}_0 : \check{Y}_0 \to B_0. \tag{2}$$

According to Conjecture 1.1 one expects to recover the mirror of Y as a compactification of $\check{Y}_0$, obtained by means of gluing on suitable singular fibres. This method raises a number of issues demanding careful consideration. As pointed out in [21, §5], understanding the structure of the singular fibres is probably one of the crucial issues.

Conjecture 1.1 appears to be the right approach if one pays attention to the topology only, i.e. forgetting about the complex and symplectic structures and considers Y and $\check{Y}$ as C^∞ manifolds only. Under some mild assumptions on the singular fibres– they are assumed to be semi-stable, i.e., with unipotent monodromy –the SYZ duality explains a topological version of Mirror Symmetry for the quintic:

Theorem 1.2 (Gross [8]). *Let $Q \subseteq \mathbb{P}^4$ be a smooth quintic 3-fold and Ξ a 4-simplex. There is a T^3 fibration $g : Q \to \partial\Xi$ with semi-stable fibres. The dual $\check{g} : \check{Q} \to \partial\Xi$ has only semi-stable fibres and $\check{Q}$ is diffeomorphic to a mirror pair of Q.*

Both fibrations g and $\check{g}$ have the same discriminant locus which consists of a trivalent graph Γ lying over the faces of $\partial\Xi$. There are three types of singular fibres present in both Q and $\check{Q}$. Let $s \in \Gamma$ and let Q_s be a singular fibre of either g or $\check{g}$. Let (b_1, b_2) where $b_i = rank H^i(Q_s, \mathbb{Z})$, $i = 1, 2$. Then Q_s can be one of the following types:

- *type* $(2, 2)$. This fibre is $S^1 \times I_1$, where I_1 is a Kodaira type I_1 fibre (a pinched torus). So, fibres of type $(2, 2)$ are singular along a circle. Fibres of type $(2, 2)$ lie over the edges of Γ;
- *type* $(1, 2)$. This fibre is obtained by collapsing a torus $T^2 \times \{p\}$ on $T^2 \times S^1$ to a point. Fibres of this kind lie over some vertices of Γ;
- *type* $(2, 1)$. Let $S \subset T^2$ be a "figure eight" (c.f. [8, fig. 2.2]). This fibre is obtained by collapsing the circles $\{p\} \times S^1$, $p \in S$, on $T^2 \times S^1$ to a point. Fibres of this kind lie over some vertices of Γ.

The $(2, 1)$ fibre is dual to the $(1, 2)$ fibre (their local monodromy representations are dual), whereas the $(2, 2)$ fibre is self-dual.

One can try to add on structures to the above topological picture. First, one can try to put suitable symplectic structures on Q and $\check{Q}$ making g and $\check{g}$ into Lagrangian fibrations. The next step would be to put suitable (almost) Calabi-Yau structures on Q and $\check{Q}$ making g and $\check{g}$ into special Lagrangian fibrations. Recent development on special Lagrangian geometry (c.f. Joyce [15]) suggests that this program may not be fully completed in the strong terms of Conjecture 1.1. This is not conclusive, however.

There has been some progress in the symplectic category. Wei-Dong Ruan [18], [20] constructs Lagrangian torus fibrations on the quintic. Ruan's method consists, roughly speaking, on a certain gradient flow deformation of a well known Lagrangian fibration on the normal crossing quintic to a neighbour non-singular quintic. This produces a piecewise C^∞ fibration with codimension 1 discriminant locus. The topology of this fibration differs from the topological T^3 fibration in [8]. Ruan argues [19] that the codimension 1 discriminant can be deformed to codimension 2, in which case the resulting fibration coincides, topologically, with the one in [8].

In this paper we are interested in the *semi-global* symplectic geometry in a neighbourhood of the singular fibres rather than in the global picture. We follow the spirit of [8] and construct singular local models of Lagrangian T^3 fibrations first. We are able to construct C^∞ Lagrangian T^3 fibrations with singular fibres resembling the topology of the $(2, 2)$ and $(1, 2)$ fibres. To date there is no symplectic model for the $(2, 1)$ fibre and it is not yet clear whether there exists a Lagrangian T^3 fibration on the quintic presenting singular fibres of type $(2, 1)$.

In two dimensions, it is known that the Kodaira type I_1 degeneration of an elliptic fibration, has an infinite number of symplectic invariants (c.f. Vũ-Ngoc [22] and [1] for an alternative approach and for the C^k-symplectic case). We show that a similar behaviour appears in dimension $n \geq 3$: the families of Lagrangian T^n fibrations considered in this paper have infinite dimensional classifying spaces.

Gross and Wilson [11] and, independently, Kontsevich and Soibelman [16] propose an alternative interpretation of SYZ Conjecture, which can be regarded as a relaxed version of Conjecture 1.1. This new proposal posits that the SYZ Conjecture is true in certain limiting sense as the mirror pair of Calabi-Yau manifolds approach large complex structure limits. The SYZ duality is then interpreted as a certain kind of Legendre transform between (singular) affine structures

on the bases of the fibrations. This conjecture is proved for K3 surfaces by Gross and Wilson [11]. For the case of the quintic, Gross constructs an affine structure on the complement of the discriminant locus (c.f. [9, §19.3]). This affine structure, in turn, induces a symplectic structure on the complement of the union of the singular fibres. The results here can be interpreted as a description of how this affine structure may become singular at the discriminant locus and the symplectic invariants arising from this degeneration.

Statement of the main results

Let $g : Q \to \partial\Xi$ be the T^3 fibration on the quintic as in Theorem 1.2. Let $s_0 \in \Gamma$ and consider $U \subseteq \partial\Xi$ an open neighbourhood of s_0. Assume U is small enough so that it contains at most one vertex of Γ and such that $\Gamma \cap U$ is connected. Then $g^{-1}(U) \subseteq Q$ is a neighbourhood of the fibre $g^{-1}(s_0)$ and the restriction of g to $g^{-1}(U)$ gives a T^3 fibration, $g^{-1}(U) \to U$, which is singular along $\Gamma \cap U$.

Now suppose there is a non-compact symplectic manifold (X, ω) together with a proper Lagrangian fibration $f : X \to B$. In addition, suppose there are diffeomorphisms Φ and ϕ giving a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "f"'] & g^{-1}(U) \arrow[d] \\ B \arrow[r, "\phi"] & U \end{tikzcd} \tag{3}$$

Let $\Delta = \phi^{-1}(\Gamma \cap U)$ and $b_0 := \phi^{-1}(s_0) \in \Delta$. Then f is a Lagrangian T^3 fibration having Δ as discriminant locus and the fibre $X_{b_0} := f^{-1}(b_0)$ is homeomorphic to the fibre $g^{-1}(s_0)$.

Definition 1.3. Let $\mathcal{L}(X_{b_0})$ denote the set of triples $\mathcal{F} = (X, \omega, f)$ where $f : (X, \omega) \to B$ is a Lagrangian fibration arising as in diagram (3). We say that two elements (X, ω, f) and (X', ω', f') in $\mathcal{L}(X_{b_0})$ are *symplectically equivalent* if there is a symplectomorphism $\Psi : X \to X'$ and a diffeomorphism $\psi : B \to B$ such that $\psi(b_0) = b_0$ and $f' \circ \Psi = \psi \circ f$. The set of equivalence classes under this relation will be denoted by $\widetilde{\mathcal{L}}(X_{b_0})$. The elements of $\widetilde{\mathcal{L}}(X_{b_0})$ can be regarded as germs of Lagrangian fibrations around the singular fibre X_{b_0}.

There are three families to be considered: $\mathcal{L}(2, 2)$, $\mathcal{L}(1, 2)$ and $\mathcal{L}(2, 1)$ corresponding to X_{b_0} of type $(2, 2)$, $(1, 2)$ and $(2, 1)$ respectively. Their discriminant loci are as depicted in Figure 1.

(a) $\kappa = 2$ (b) $\kappa = 1$

Figure 1: The discriminant loci $\Delta \subset B$.

Theorem 1.4. *There is an element $\mathcal{F}_H \in \mathcal{L}(\kappa, 2)$, $\kappa = 1, 2$, for each $H \in C^\infty(B)$.*

Theorem 1.5. *The germs of fibrations of type $\widetilde{\mathcal{L}}(\kappa, 2)$, $\kappa = 1, 2$, are classified by $C^\infty_\Delta(B)$, the space of germs of C^∞ functions on B vanishing at Δ to all orders.*

2 Preliminaries

We recall the construction of action-angle variables on Lagrangian T^n bundles. This is an extensively used technique in the context of Hamiltonian mechanics. The material presented here is standard (c.f. [2] and [6, §2]).

Let (X, ω) be a symplectic manifold of dimension $2n$ and B a n-dimensional manifold. We shall assume X and B to be connected but not necessarily compact.

Definition 2.1. Let $f : X \to B$ be a proper C^∞ map with connected fibres and denote by $Crit(f) \subset X$ the set of points in X where the differential f_* is not surjective. Let $X^\# = X \setminus Crit(f)$ and $f^\#$ denote the restriction of f to $X^\#$. If the fibres of $f^\#$ are Lagrangian with respect to ω we say that f is a *Lagrangian fibration*. We denote a Lagrangian fibration as a triple $\mathcal{F} = (X, \omega, f)$.

Observe that the Arnold-Liouville theorem implies that the regular fibres of $\mathcal{F}$ as in Definition 2.1 are diffeomorphic to T^n.

Definition 2.2. Let $\mathcal{F} = (X, \omega, f)$ be a Lagrangian fibration. Denote $X_b = f^{-1}(b)$ and let $Crit(X_b) = Crit(f) \cap f^{-1}(b)$ the set of singular points of X_b. We say that $\mathcal{F}$ is *admissible* if

(1) $Crit(X_b)$ is connected and the fibres of $f^\#$ are connected;

(2) $\Delta = f(Crit(f))$ is a closed codimension two subset of B and

(3) $f^\#(X^\#) = B$ and for any point $x \in X^\#$ there is a local C^∞ section of f passing through x.

Observe that (3) in Definition 2.2 implies that f does not have singular fibres dropping dimension. The fibres of $f^\#$ over Δ are diffeomorphic to $T^k \times \mathbb{R}^{n-k}$. From now on we only consider Lagrangian fibrations which are admissible.

Now let $B_0 = B \setminus \Delta$, $X_0 = f^{-1}(B_0)$ and $f_0 = f|_{X_0}$. The map, $f_0 : X_0 \to B_0$ defines a T^n fibre bundle, denoted by (X_0, f_0). Consider $R^{n-1}f_{0*}\mathbb{Z}$, a local system as defined in §1. Since f_0 is proper, one can identify the stalk $(R^{n-1}f_{0*}\mathbb{Z})_b$ with $H_1(X_b, \mathbb{Z})$ using Poincaré duality.

Now consider $f^\# : X^\# \to B$ and let $X_b^\# = f^{\#-1}(b)$. We define a sheaf on B, with stalk[1] $H_c^i(X_b^\#, \mathbb{Z})$ as follows. Let $U \subseteq B$ and consider the presheaf defined by $U \mapsto H_c^i(f^{\#-1}(U), \mathbb{Z})$. The latter induces a sheaf, denoted $R_c^i f_*^\# \mathbb{Z}$, with stalk $(R_c^i f_*^\# \mathbb{Z})_b \cong H_c^i(X_b^\#, \mathbb{Z})$. Again, we can identify $H_r(X_b^\#, \mathbb{Z})$ with $(R_c^{n-r} f_*^\# \mathbb{Z})_b$.

Now we define a map $R_c^{n-1} f_*^\# \mathbb{Z} \hookrightarrow T_B^*$ as follows. For each $U \subseteq B$ open and $b \in U$ let $\gamma(b) \in H_1(X_b^\#, \mathbb{Z}) \cong H_c^{n-1}(X_b^\#, \mathbb{Z})$, $v \in T_{U,b}$ and $\tilde{v}$ a lifting of v. Define the map $(b, \gamma(b)) \mapsto \lambda_b$, where

$$\lambda_b(v) = -\int_{\gamma(b)} \iota(\tilde{v})\omega. \tag{4}$$

This gives a local section $b \mapsto \lambda_b$ of T_B^*, i.e. a 1-form on $U \subseteq B$. One can check that the above formula does not depend on the lifting of v.

Definition 2.3. Let $\Lambda \subset T_B^*$ be the image of $R_c^{n-1} f_*^\# \mathbb{Z}$ under the map (4). We call Λ the *period lattice* of f.

Now let us consider $R^{n-1}f_{0*}\mathbb{Z}$. Choose a local section γ of $R^{n-1}f_{0*}\mathbb{Z}$ over an open set on $U \subseteq B_0$. The image of this section under the map (4) gives us a period 1-form λ_γ. This form is closed, since it is the differential of the action function:

$$\mathcal{A}_\gamma(b) = \int_{\gamma(b)} \sigma. \tag{5}$$

Here σ is such that $d\sigma = \omega$. We can ensure that such a σ always exists on $f_0^{-1}(U)$ by taking $U \subset B_0$ small enough. This means that the sections of Λ are given locally by the image of a closed 1-form and, in particular, Λ is Lagrangian with respect to the canonical symplectic structure on T_B^*.

The above construction gives us an exact sequence:

$$0 \longrightarrow R_c^{n-1} f_*^\# \mathbb{Z} \longrightarrow T_B^* \longrightarrow T_B^*/\Lambda \longrightarrow 0 \tag{6}$$

Proposition-Definition 2.4. The sequence (6) defines a symplectic manifold, $J^\# := T_B^*/\Lambda$, and a Lagrangian fibration $\mathcal{J}_f : J^\# \to B$ with fibre $\mathcal{J}_f^{-1}(b) = T_{B,b}^*/\Lambda_b$. We call $\mathcal{J}_f$ the *Jacobian fibration* of f.

[1]Here $H_c^*(\,\cdot\,, \mathbb{Z})$ denotes compactly supported cohomology with coefficients in $\mathbb{Z}$

Proof. The Lagrangian nature of $\Lambda \subset T^*_B$ implies that translations over Λ along the fibres of T^*_B are symplectic transformations. Therefore, $J^\# := T^*_B/\Lambda$ inherits the canonical symplectic structure of T^*_B. The bundle projection $T^*_B \to B$ induces the map $\mathcal{J}_f : J^\# \to B$. It follows immediately that the fibres of $\mathcal{J}_f$ are Lagrangian. $\square$

The following result is deduced from [2] (c.f. [6, §2]):

Theorem 2.5. *Let (X, ω, f) be a proper Lagrangian fibration. Let $\mathcal{J}_f : J^\# \to B$ be the Jacobian fibration of f as defined in (2.4). Then,*

(i) if $f^\#$ has a global section, $\Sigma : B \to X^\#$, then there is a fibre preserving diffeomorphism $\Psi : J^\# \to X^\#$;

(ii) if Σ is Lagrangian, then the diffeomorphism in (i) is a symplectomorphism.

Duistermaat [2] observed that a Lagrangian T^n bundle $f_0 : X_0 \to B_0$ has three invariants: its monodromy, its Chern class and $[\omega] \in H^2(X_0, \mathbb{R})$. This tells us that, by taking $U \subset B_0$ contractible, we can define a set of action-angle (canonical) coordinates on $f^{-1}(U) \subset X$ which allows us to write ω on $f^{-1}(U)$ as the standard symplectic structure. Furthermore, the action coordinates provide B_0 with an integral affine structure.

3 The family $\mathcal{L}(2, 2)$

Lagrangian T^2-fibrations with singular fibre of type I_1 are better known in symplectic geometry as *focus-focus* singularities; they appear in a number of "physically relevant" integrable Hamiltonian systems.

Theorem 3.1 ([1]). *Let $D \subseteq \mathbb{C}$ be an open disk with coordinates $s = s_1 + \sqrt{-1}s_2$. For any function $h \in C^\infty(D)$ there is a Lagrangian T^2 fibration $\mathcal{F} = (\bar{X}, \omega, f)$ with singular fibre of focus-focus type and whose period lattice is generated by $\tau_1 = -\log|s|ds_1 + \mathrm{Arg}(s)ds_2 + dh$ and $\tau_2 = 2\pi ds_2$.*

Remark 3.2. There is an alternative proof of Theorem 3.1 proposed by Vũ-Ngoc [22].

Proposition 3.3. *Let $\bar{f} : \bar{X} \to D$ be a T^2 fibration as in Theorem 3.1 and $(0, 1)$ an open interval. Let $X = \bar{X} \times S^1 \times (0, 1)$ and define $f : X \to D \times (0, 1)$ to be the composition of the projection onto $\bar{X} \times (0, 1)$ and $\bar{f} \times id$. Then, there is a symplectic structure ω on X making the fibres of f Lagrangian. Furthermore the fibres over $\Delta = \{0\} \times (0, 1)$ are diffeomorphic to $I_1 \times S^1$.*

Proof. Let (r, θ) be coordinates on $(0, 1) \times S^1$. Define $\omega = \bar{\omega} + dr \wedge d\theta$. One can verify f is Lagrangian with respect to ω. $\square$

Definition 3.4. Let M be a symplectic manifold and let $f_1, \ldots, f_n \in C^\infty(M)$ define an integrable Hamiltonian system on M. Let $x \in M$ and let $(t_j; x) \mapsto \phi_j^{t_j}(x)$ be the flow generated by the Hamiltonian vector field of f_j. We call $(t_1, \ldots, t_n; x) \mapsto \phi_n^{t_n} \circ \cdots \circ \phi_1^{t_1}(x)$ the *Poisson action* of the system. If all flows ϕ_j are complete the Poisson action is an $\mathbb{R}^n$-action on M which preserves the fibres of the map $x \mapsto (f_1(x), \ldots, f_n(x))$.

Observe that for f as in Proposition 3.3 all points $x \in Crit(f)$ are non-degenerate and such that rank $f_*|_x = 1$. Regarding f as an integrable system, one can check that the Poisson orbit of x, $\mathcal{O}_x$, is diffeomorphic to S^1 and each point in $\mathcal{O}_x$ is a rank one critical point. Rank one singular orbits of integrable systems are classified up to fibre preserving symplectomorphism. We state here a special case of a result due to Miranda and Tien-Zung [17].

Theorem 3.5 ([17]). *Let (M^6, Ω, h) be a (not necessarily proper) Lagrangian fibration with a non-degenerate rank 1 singular orbit $\mathcal{O}$ of the Poisson action. Let D^4 be a 4-ball and let $V = D^4 \times (-1, 1) \times S^1$ be a symplectic 6-manifold with canonical coordinates (x_j, y_j, r, θ). There exists a neighbourhood $U \subseteq M$ of $\mathcal{O}$ a Lagrangian fibration, $L : V \to D \times (-1, 1)$, $L(x_j, y_j, r, \theta) = (q_1(x_j, y_j), q_2(x_j, y_j), r)$, and a fibre preserving symplectomorphism $\psi : U \to V$ sending $\mathcal{O}$ to $\{x_i = y_i = r = 0\}$ and such that q_i can be one of the following types:*

$$\begin{array}{ll} \textit{elliptic type:} & q_i = x_i^2 + y_i^2 \\ \textit{hyperbolic type:} & q_i = x_i y_i \\ \textit{focus-focus type:} & \begin{cases} q_i = x_i y_i + x_{i+1} y_{i+1} \\ q_{i+1} = x_i y_{i+1} - x_{i+1} y_i \end{cases} \end{array}$$

Definition 3.6. Let $\mathcal{F} = (X, \omega, f)$ be an admissible Lagrangian T^3 fibration. Let $x \in Crit(X)$ be a non-degenerate rank 1 singular point and $\mathcal{O}_x$ its Poisson orbit. We say that $\mathcal{F}$ is a *Lagrangian fibration of type* $(2, 2)$, denoted $\mathcal{F} \in \mathcal{L}(2, 2)$, if there is a neighbourhood $U \subset X$ of $\mathcal{O}_x$ such that $X = f^{-1}(f(U))$ and the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "f|_U"'] \arrow[dr, "F"] & V \arrow[d, "q=(q_1,q_2,q_3)"] \\ f(U) \arrow[r, "\phi"] & D \times (0,1) \end{tikzcd} \tag{7}$$

where $\psi : U \to V = D^4 \times (0, 1) \times S^1$ is a symplectomorphism, ϕ is a diffeomorphism and q_1, q_2 are of focus-focus type, $q_3 = r$.

We shall denote $B = D \times (0, 1)$ and $b = (b_1, b_2, b_3) \in B$. We can write $\phi = (\phi_1, \phi_2, \phi_3)$ and $\phi \circ f = (f_1, f_2, f_3)$, where $f_i = \phi_i \circ f$. If we think of ψ as providing U with canonical coordinates, then $f_j|_U = q_j$ or, with slight abuse of notation, $f|_U = F$ where $F(x_i, y_i) = (q_1, q_2, q_3)$ as in (7). We regard F as the normal form for the family $\mathcal{L}(2, 2)$.

Let v_{q_j} be the Hamiltonian vector field corresponding to q_j and let g_j^t its flow. Let $\zeta_1 = x_1 + \sqrt{-1}x_2$ and $\zeta_2 = y_1 + \sqrt{-1}y_2$. Observe that $\bar{\zeta}_1\zeta_2 = q_1 + \sqrt{-1}q_2$. The flows of $g_j : \mathbb{R} \times V \to V$ are given by:

$$\begin{aligned} g_1^t(\zeta_1, \zeta_2, r, \theta) &= (e^t\zeta_1, e^{-t}\zeta_2, r, \theta) \\ g_2^t(\zeta_1, \zeta_2, r, \theta) &= (e^{it}\zeta_1, e^{it}\zeta_2, r, \theta) \\ g_3^t(\zeta_1, \zeta_2, r, \theta) &= (\zeta_1, \zeta_2, r, \theta - t). \end{aligned} \tag{8}$$

Observe that g_2^t and g_3^t generate a fibre-preserving T^2 action on V.

Lemma 3.7. *Let $(X, \omega, f) \in \mathcal{L}(2, 2)$. Then the compact fibres of $f^\# : X^\# \to B$ are diffeomorphic to T^3 whereas the non-compact ones are diffeomorphic to $T^2 \times \mathbb{R}$. There is an open neighbourhood $\mathcal{U} \subset X$ of $Crit(f)$ such that the fibres of $f_\mathcal{U} := f|_{X \setminus \mathcal{U}}$ are diffeomorphic to $T^2 \times [0, 1]$. Furthermore, $f_\mathcal{U}$ defines a trivial fibre bundle.*

Proof. The first part follows directly from the definition. For the second claim it is enough to take $\mathcal{U}$ a small connected neighbourhood of $Crit(f)$ which is invariant with respect to the T^2-action induced by v_{q_2} and v_{q_3} and redefine $B := \phi \circ f(\mathcal{U})$. The triviality of $f_\mathcal{U}$ follows from the fact that B is contractible. □

Notice that $f_j|_\mathcal{U} = q_j$ implies that the vector fields v_{q_j} extend vector Hamiltonian fields v_j on X which are tangent to the fibres, hence the flows g_i^t extend to X. Since the fibres of f are compact g_i^t are complete.

Construction 3.8. Define an action $\Pi : \mathbb{R}^3 \times X \to X$, $(T, x) \mapsto \Pi^T(x)$, $T = (t_1, t_2, t_3)$ as the composition of flows:

$$\Pi(T, x) := g_3^{t_3} \circ g_2^{t_2} \circ g_1^{t_1}(x).$$

The restriction of Π to $X^\#$ is just the Poisson action on $f^\# : X^\# \to B$, which is free and transitive along the fibres since $f^{\#-1}(b)$ is connected. This implies that for any two points $x, y \in f^{\#-1}(b)$, there is a multi-time $T = T(x, y) \in \mathbb{R}^3$ such that $\Pi^T(x) = y$. Similarly, consider now the Hamiltonian vector fields $v_{q_1}, v_{q_2}, v_{q_3}$, on $\mathcal{U} \subseteq X$. In an analogous way, we can define an action on $\mathcal{U}$, $\Pi_0 : R \times \mathcal{U} \to \mathcal{U}$, where $R \subseteq \mathbb{R}^3$ is some open set, as the composition of flows of v_{q_i}. Since $F^{-1}(b)$ is connected and non-singular for $b \in B_0 = B \setminus \Delta$, Π_0 is transitive along the regular fibres of F.

We are going to use the actions Π and Π_0 to compute the period lattice of (X, ω, f). Let $\epsilon > 0$ and write $s = b_1 + \sqrt{-1}b_2$, $r = b_3$. Define, $\Sigma_1(s, r) = (\bar{s}/\epsilon, \epsilon, r, \theta_0)$ and $\Sigma_2(s, r) = (\epsilon, s/\epsilon, r, \theta_0)$,

$\theta_0 \in S^1 = \mathbb{R}/\mathbb{Z}$. These give sections of F which lie inside $\mathcal{U}$ and do not intersect $Crit(F)$. Now consider the equation:

$$\Pi_0(T_0(b), \Sigma_1(b)) = \Sigma_2(b), b \in B_0. \tag{9}$$

The solution $T_0 = (\alpha_1, \alpha_2, \alpha_3)$ is determined by the system:

$$\begin{cases} e^{-\alpha_1 + i\alpha_2} \cdot \epsilon = s/\epsilon \\ e^{\alpha_1 + i\alpha_2} \cdot \bar{s}/\epsilon = \epsilon \\ \theta_0 - \alpha_3 = \theta_0 \end{cases}$$

One verifies that the (primitive) solution to the system is

$$\alpha_1 = -\log|s| + 2\log\epsilon, \quad \alpha_2 = \mathrm{Arg}(s), \quad \alpha_3 = 0.$$

Let $\mathcal{U}'$ be a T^2 invariant neighbourhood of $Crit(f)$, $\mathcal{U}' \subset \mathcal{U}$ as in Lemma 3.7. We can take $\mathcal{U}'$ small enough so that we can regard Σ_1 and Σ_2 also as sections of the $T^2 \times \mathbb{R}$ fibre bundle $f_{\mathcal{U}'}$ over B.

Proposition 3.9. *Let Σ_1 and Σ_2 be sections of $f_{\mathcal{U}'}$ as above. The equation:*

$$\Pi^{T(b)}(\Sigma_2(b)) = \Sigma_1(b), \qquad b \in B \tag{10}$$

has a unique solution, $T(b) = (\eta_1(b), \eta_2(b), \eta_3(b))$, which depends smoothly on $b \in B$.

Proof. A solution to equation (10) exists since the action Π is transitive along the fibres of $f_{\mathcal{U}'}$. We shall see that $T(b)$ depends smoothly on $b \in B$.

Let $S_j = \Sigma_j(B)$ and let $x_0 \in S_2$ such that $f(x_0) = b_0 \in B$. Then, there is $t_0 \in \mathbb{R}^3$ such that $\Pi^{t_0}(x_0) = y_0 \in S_1$. Let U_0 be a small neighbourhood of x_0 and let R be a neighbourhood of t_0. Let V_0 be a neighbourhood of y_0 such that $f(V_0) = f(U_0) \subseteq B$. Define $P : R \times (U_0 \cap S_2) \to V_0$, as $P(t, x) = \Pi^t(x)$. Notice that S_1 is transversal to the $\mathbb{R}^3$-orbit of Π passing through a point $y \in S_1$. This implies that P is transversal to $S_1 \cap V_0$. Then, $\mathcal{P} := P^{-1}(S_1 \cap V_0)$ is a codimension 3 smooth submanifold of $R \times (U_0 \cap S_2)$.

Now observe that since Π is an action, the "time" derivative of P evaluated at (t_0, x_0) is non-singular. Then, $\mathcal{P}$ can be described locally as the graph of a C^∞ map, $g : U_0' \to R$, where $U_0' \subseteq (S_2 \cap U_0)$ is a small neighbourhood of x_0. Let $B' = f(U_0')$ and define $T : B' \subseteq B \to R$, as $T(b) = g \circ \Sigma_2(b)$ for $b \in B'$. Then, T is a $\mathbb{R}^3$-valued C^∞ function such that $\Pi^{T(b)}(\Sigma_2(b)) = \Sigma_1(b)$. From Lemma 3.7 we know that $f_{\mathcal{U}'}$ has trivial monodromy. Therefore these local solutions can be glued together to give a single-valued global solution, $T(b) = (\eta_1(b), \eta_2(b), \eta_3(b))$, $b \in B$. $\square$

Define the 1-form $\eta = \eta_1 db_1 + \eta_2 db_2 + \eta_3 db_3$ on B where $\eta_i \in C^\infty(B)$ are as in Proposition 3.9.

Proposition 3.10. *Let $\mathcal{F} = (X, \omega, f) \in \mathcal{L}(2, 2)$. There are local sections (e_1, e_2, e_3) of $R_c^2 f^{\#}\mathbb{Z}$ such that the period lattice of $\mathcal{F}$ is generated by the 1-forms:*

$$\tau_1 = \tau_0 + dH, \quad \tau_2 = 2\pi ds_2, \quad \tau_3 = dr$$

where $\tau_0 = -\log|s|ds_1 + \mathrm{Arg}(s)ds_2$ and H is a smooth function of $b = (s_1, s_2, r) \in D \times I$ such that $dH = \eta$. The monodromy of f expressed in terms of $\Lambda = \langle \tau_1, \tau_2, \tau_3 \rangle$ is represented by the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof. One can construct generators $e_1(b), e_2(b), e_3(b)$ of $H_1(X_b, \mathbb{Z})$, $b \in B_0$ by means joining integral curves of v_j in a suitable way. For instance, we define a representative of e_1 to be the ordered composition of paths $\gamma = (\gamma_1, \gamma_2, \gamma_3, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$. Here, γ_i is an integral curve of v_{q_i} starting at a point x_{i-1} running a time $t_i \in [0, \alpha_i]$ and finishing at a point x_i. Similarly, $\tilde{\gamma}_i$ is an integral curve of v_i starting at a point $\tilde{x}_{i-1}$ running a time $\tilde{t}_i \in [0, \eta_i]$ and finishing at a point $\tilde{x}_i$. Then, the curve γ is determined by the initial condition $x_0 = \Sigma_1(b)$, $\tilde{x}_0 = x_3$. It follows from equations (9) and (10) that γ is closed and non-trivial. For constructing a representative of e_j, $j = 2, 3$, we take an integral curve of v_j starting at $\Sigma_1(b)$ and flowing from time 0 to 1. Now we can use formula (4) to compute the period 1-forms. It follows that $\tau_1 = \sum \alpha_j db_j + \sum \eta_j db_j$. Since τ_1 and $\tau_0 = -\log|s|ds_1 + \mathrm{Arg}(s)ds_2$ are closed, then $\tau_1 = \tau_0 + dH$ for some $H \in C^\infty(B)$. The computation of τ_2 and τ_3 is direct from (4). $\square$

4 The family $\mathcal{L}(1,2)$

There is a fairly complete understanding of the class of non-degenerate (Morse-Bott) singularities of integrable Hamiltonian systems (cf. Eliasson [3], Tien-Zung and Miranda [17]). Generically, the function components of the fibration– i.e. the integrals of the system –can be reduced to quadratic polynomials. In contrast, for some special Lagrangian singularities arising from integrable Hamiltonian systems, one should expect cubic terms (c.f. Fu [4]).

T^2-symmetric special Lagrangian singularities

Let X be a symplectic 6-manifold and $f:(X,\omega)\to B$ a Lagrangian fibration which is admissible in the sense of Definition 2.2. Denote by $\omega_0=\sum_i dx_i\wedge dy_i$ the standard symplectic form on $\mathbb{C}^3\cong\mathbb{R}^6$ with canonical coordinates (x_i,y_i), $z_i=x_i+\sqrt{-1}y_i$ and let $\Omega_0=dz_1\wedge dz_2\wedge dz_3$.

Definition 4.1. Let $f:(X,\omega)\to B$ be a Lagrangian fibration and let $p\in Crit(f)$ and let $k=rank f_*|_p$. Let O_p denote the Poisson orbit of p. We say that O_p is a *rank k complexity one singularity* if there is an open neighbourhood $W\subseteq X$ of O_p and a Hamiltonian T^2 action $\Phi:T^2\times W\to W$ such that $f(\Phi(t,x))=f(x)$ for each $(t,x)\in T^2\times W$.

Remark 4.2. Regarding $f|_W$ as an integrable Hamiltonian system, we can think of T^2 as a symmetry group of the system. Notice that if $k=0$, i.e. p is a fixed point of the Poisson action, then p is also a fixed point of Φ. It is a standard fact that the Hamiltonian action of a k-torus on a symplectic manifold M is completely determined on a neighbourhood of a fixed point x_0 by the *weights* of the isotropy representation of the linear action of T^k on $T_{x_0}M$. These are elements $\rho_1(x_0),\dots\rho_n(x_0)\in\mathfrak{t}^*=Lie(T^k)^*$ (c.f. Guillemin and Sternberg [12]).

Definition 4.3. Let $f:(X,\omega)\to B$ be a Lagrangian fibration, $p\in Crit(f)$ and $O_p\in X$ a rank k Poisson orbit of f. We say that O_p is *special* if there is an open neighbourhood $U\subseteq X$ of O_p and canonical coordinates $(z_i,\bar{z}_i)$ on U such that $f|_U$ is special Lagrangian with respect to (ω_0,Ω_0).

Example 4.4. Let $(X,\omega,f)\in\mathcal{L}(2,2)$ and $p\in Crit(f)$. Then $O_p\subset X$ is a special singularity, which is also a rank one complexity one singularity.

The Harvey-Lawson singularity

We review an example proposed by Harvey and Lawson (c.f. [13, §III.3.A]). This provides an example of a special rank zero complexity one singularity.

Let us consider the map $F:\mathbb{C}^n\to\mathbb{R}^n$ given by $F=(F_1,\dots,F_n)$ where

$$F_1=\operatorname{Im}\prod z_i,\;\; F_k=|z_1|^2-|z_k|^2,\quad k=2,\dots n. \tag{11}$$

The fibres of F are Lagrangian with respect to the standard symplectic form on $\mathbb{C}^n$; for this, one only needs to check that $\{F_i,F_j\}=0$ for $i,j=1,\dots,n$. In other words, F defines an integrable Hamiltonian system. One can also check that $\operatorname{Re}(\det_{\mathbb{C}}(\partial_{\bar{z}_j}F_i))=0$, hence the fibres of F are special Lagrangian. We observe that the map $\mu:=(F_2,\dots,F_n)$ is the moment map of the Hamiltonian T^{n-1} action on $\mathbb{C}^n$ given by:

$$(z_1,\dots,z_n)\mapsto(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n),$$

with $\theta_1+\dots+\theta_n=0$. This action preserves the fibres of F. Now let $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and let $z\in F^{-1}(x)$. Denote by $T\cdot z$ the T^{n-1}-orbit of z. Then, $T\cdot z$ is homeomorphic to T^{n-1} unless $z\in Crit(F)=\bigcup_{1\le i<j\le n}P_{ij}$ where,

$$P_{ij}=\{(z_1,\dots,z_n)\in\mathbb{C}^n\mid z_i=z_j=0\}. \tag{12}$$

For $z\in Crit(F)$, the orbit $T\cdot z$ is a torus of lower dimension and it is a point when $z=0$. A fibre $F^{-1}(x)$ disjoint from $Crit(F)$ is homeomorphic to $T^{n-1}\times\mathbb{R}$ and for $x\in\Delta:=F(Crit(F))$ the fibre $F^{-1}(x)$ is a singular fibre.

For $n=3$, $\Delta=\Delta_1\cup\Delta_2\cup\Delta_3\cup\{0\}$ where $\Delta_1=\{x_1=0,x_2=x_3>0\}$, $\Delta_j=\{x_1=x_j=0,x_j<0\}$, for $j=2,3$. The fibre over $x\in\Delta_i$ is homeomorphic to

$$S^1\times[\mathbb{R}\times S^1/(\{point\}\times S^1)],$$

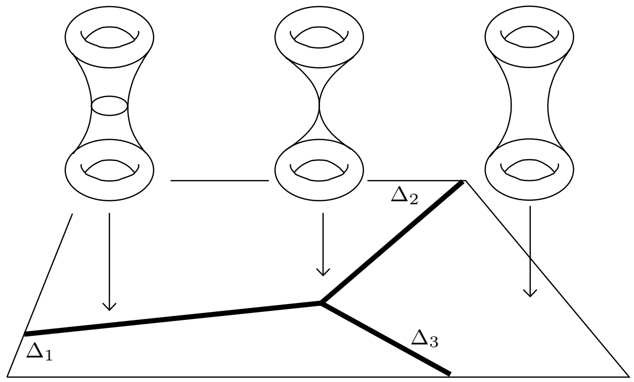

Figure 2: The fibres of F

whereas the fibre over $0 \in \mathbb{R}^3$ is homeomorphic to

$$\mathbb{R} \times T^2/(\{point\} \times T^2)$$

In particular, we conclude that the map F is not proper.

Remark 4.5. Joyce observed (c.f. [14, §5] and [15, §4]) that, in three dimensions, any connected special Lagrangian 3-manifold in $\mathbb{C}^3$ which is invariant under the above T^2-action is a subset of some fibre of the map (11).

The topological $(1,2)$ fibre

We outline Gross' construction of a topological 3-torus fibration with fibre of type $(1,2)$. For the details we refer the reader to [8, Example 2.10].

Construction 4.6 (Gross [8]). Let $B = B^3$ be a 3-ball. We define $\Delta \subset B$ a cone over three points as follows. Identify $B \setminus \{0\}$ with $S^2 \times (0,1)$ and let $p_1, p_2, p_3 \in S^2$. Define $\Delta_i = \{p_i\} \times (0,1)$. These are the "legs" of the cone. Define $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \{0\}$, where $\{0\}$ is the vertex of the cone.

Let $Y = S^1 \times B$ and $Y' = Y \setminus (\{p\} \times \Delta)$, where $p \in S^1$. Let $L \cong \mathbb{Z}^2$ and define $T(L) = L \otimes_{\mathbb{Z}} \mathbb{R}/L$. Now consider a principal $T(L)$-bundle $\pi' : X' \to Y'$ with Chern class $c_1 \in H^2(Y', L)$. Then the class c_1 is represented by a triple (a_1, a_2, a_3) where $a_i \in L$. It is shown (c.f. [8, Ex. 2.10]) that by choosing $c_1 = ((1,0),(0,1),(-1,-1))$ there is a unique manifold X such that $X' \subset X$ and a commutative diagram of smooth maps:

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, hook] & Y \end{tikzcd}$$

such that π is proper. Furthermore, it is shown that in a neighbourhood $U \cong \mathbb{C} \times \mathbb{R}^2 \subset Y$ of the vertex of Δ, the map $\pi : \pi^{-1}(U) \to U$ coincides with the map $\tilde{\pi} : \mathbb{C}^3 \to \mathbb{C} \times \mathbb{R}^2$ given by:

$$\tilde{\pi}(z_1, z_2, z_3) = \left(z_1 z_2 z_3, |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2\right). \tag{13}$$

Now define $f : X \to B$ to be the composition of $\pi : X \to Y$ with the projection $Y \to B$. Then, f is a continuous map whose fibre over $b \in B \setminus \Delta$ is T^3. The fibre over $b \in \Delta_i$ is homeomorphic to $S^1 \times [S^1 \times S^1/(\{point\} \times S^1)]$, i.e. it is a $(2,2)$ fibre, whereas the fibre over the vertex of Δ is homeomorphic to $S^1 \times T^2/(\{point\} \times T^2)$, i.e. it is a $(1,2)$ fibre.

It turns out that the $T(L)$ action on X' action extends to X, moreover, $Crit(f) \subset X$ consists of the union of the critical orbits of this action. There is a single fixed point $p \in Crit(f)$, which is singular point of $f^{-1}(0)$.

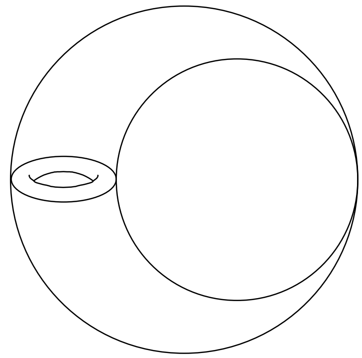

Figure 3: Singular fibre of type $(1,2)$

The symplectic structure

Let $f : X \to B$ as in Construction 4.6 and suppose there is a symplectic structure ω making f Lagrangian, i.e. defining a triple $(X, \omega, f) \in \mathfrak{L}(1,2)$. Furthermore, assume the extended $T(L)$ action on X preserves ω. It is follows from these hypotheses that p is a rank zero complexity one singularity. Let $\mathfrak{t} = Lie(T(L))$. Then we can regard $L \hookrightarrow \mathfrak{t}$ and identify c_1 with the isotropy data $(\rho_1(p), \rho_2(p), \rho_3(p))$ of the $T(L)$-action at p.

Theorem 4.7. *Let $f : (X, \omega) \to B$ be a Lagrangian fibration of type $(1,2)$. Assume there is a fibre-preserving $T := T(L)$ action preserving ω. Let $p \in Crit(f) \cap f^{-1}(0)$. Then there is an open neighbourhood $U \subset X$ of p, a symplectomorphism $\psi : U \to V \subseteq (T_pX, \omega_0)$ and a diffeomorphism $\varphi : f(U) \to \mathbb{R}^3$ such that $q \circ \psi = \varphi \circ f|_U$, where $q : V \to \mathbb{R}^3$ is a Lagrangian fibration given by $q = (h, |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2)$, $h \in C^\infty(V)$. Furthermore, if p is special, then $h = \operatorname{Im} z_1 z_2 z_3$.*

Proof. Consider $\mu : X \to \mathfrak{t}^*$ the moment map of the T action around p. According to [12], there is a neighbourhood $U \subseteq X$ of p and an equivariant symplectomorphism $\psi : U \to V \subseteq (T_pX, \omega_0)$ such that $\mu = \psi^* M$, where $M = c + \sum_i \rho_i |z_i|^2$, $c \in \mathfrak{t}^*$. Without loss of generality we can assume $c = 0$ and choose a basis of $\mathfrak{t}^*$ such that $\rho_1 = (1,1)$, $\rho_2 = (-1,0)$ and $\rho_3 = (0,-1)$. Then we can write $M = (M_1, M_2)$, where $M_j = |z_1|^2 - |z_j|^2$. Let v_j be the vector field on V determined by the equation: $\iota(v_j)\omega_0 = dM_j$. The orbits of v_j are periodic with period 2π. Now let Σ be a section of f over $B' := f(U)$ such that $\Sigma(B') \subset U \setminus Crit(f)$. Let $y(b) = \psi(\Sigma(b))$ and $g_j : [0, 2\pi] \to V$ be an integral orbit of v_j passing through $y(b)$. Then g_j pulls back to a loop $\gamma_j(b) \subset f^{-1}(b) \cap U$, disjoint from $Crit(f)$. We can assume there is a 1-form σ such that $d\sigma = -\omega$. Let $\mathcal{A}_j(b) = \int_{\gamma_j(b)} \sigma$. One can verify that $\mathcal{A}_j \circ f|_U = M_j \circ \psi$. Now let α be a Lagrangian section of $T^*_{B'}$ close to the zero section. We can choose α such that $\alpha(0) \wedge d\mathcal{A}_1(0) \wedge d\mathcal{A}_2(0) \neq 0$. Then there is an open neighbourhood of $0 \in B'$ in which $\alpha \wedge d\mathcal{A}_1 \wedge d\mathcal{A}_2 \neq 0$ and a unique smooth function $\mathcal{A}$ such that $\mathcal{A}(0) = 0$ and $d\mathcal{A} = \alpha$. Then $\varphi = (\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2)$ defines a diffeomorphism from a small neighbourhood of $0 \in \mathbb{R}^3$ denoted, with abuse of notation by B, into $\mathbb{R}^3$, $\varphi : B \to \varphi(B) \subseteq \mathbb{R}^3$. Let $h = \mathcal{A} \circ f \circ \psi^{-1}$. Then h is a T-invariant function on V hence $q := (h, M_1, M_2)$ defines a Lagrangian fibration on V such that $q = \varphi \circ f|_U \circ \psi^{-1}$. Now we can think of ψ as identifying $U \cong V \subseteq \mathbb{C}^3$ such that $\varphi \circ f|_U = (h, M_1, M_2)$. In view of Remark 4.5 and since $\psi(Crit(f)) = Crit(q) = Crit(M) = \bigcup_{ij}\{z_i = z_j = 0\}$, we see that if $\varphi \circ f|_U$ is special Lagrangian then there should exist a 1-form α with the above properties and such that $h = \operatorname{Im} z_1 z_2 z_3$. $\square$

Remark 4.8. Observe that the $T(L)$-action on X can always be assumed to be Hamiltonian. Indeed, the above action is chosen so that f has the desired monodromy, in particular, it induces monodromy invariant cycles $e_1(b), e_2(b) \in H_1(f^{-1}(b), \mathbb{Z})$ which can be used to compute the action integrals $\mathcal{A}_{e_1}$, $\mathcal{A}_{e_2}$. Then $\mu_i = \mathcal{A}_i \circ f$ define the moment map (μ_1, μ_2) of a $T = S^1 \times S^1$ action, which is defined on X as e_i are monodromy invariant. It is a consequence of [8, Prop. 3.3] and [7, Thm. 2.2] that $p \in f^{-1}(0) \cap Crit(f)$ can be made into a special singularity with respect to (ω_0, Ω_0).

Corollary 4.9. *Let $(X, \omega, f) \in \mathfrak{L}(1,2)$. Let $p \in f^{-1}(0) \cap Crit(f)$ and let (T_pX, ω_0), where $\omega_0 = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$. There is a neighbourhood $U \subset X$ of p and a 3-ball B centred at $0 \in \mathbb{R}^3$ such that $f(U) = B$, a diffeomorphism $\varphi : B \to \varphi(B) \subseteq \mathbb{R}^3$ and a symplectomorphism $\psi : U \to V \subseteq (T_pX, \omega_0)$ such that $\varphi \circ f|_U = F \circ \psi$ where $F(z_1, z_2, z_3) = (\operatorname{Im} z_1 z_2 z_3, |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2)$.*

Example of a Lagrangian fibration of type $(1, 2)$

Here we show that the family $\mathfrak{L}(1, 2)$ is not void. We construct a Lagrangian fibration of type $(1, 2)$ for each $H \in C^\infty(B)$, $B \subset \mathbb{R}^3$ an open ball. The arguments we use here are valid in any dimension.

Consider the map $F : \mathbb{C}^n \to \mathbb{R}^n$, where $F = (F_1, \ldots, F_n)$ as in (11). The quotient $\mathbb{C}^n/T^{n-1}$ can be identified with $\mathbb{C} \times \mathbb{R}^{n-1}$ by means of the map $\pi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{R}^{n-1}$,

$$\pi(z) = (\prod z_i, |z_1|^2 - |z_2|^2, \ldots, |z_1|^2 - |z_n|^2). \tag{14}$$

Let $\prod z_i = u + \sqrt{-1}b_1 \in \mathbb{C}$ and $b_j = |z_1|^2 - |z_j|^2$ and $b = (b_1, \ldots b_n)$. Letting $x_i = |z_i|^2$ the following relations hold:

$$\begin{cases} \prod_{i=1}^n x_i = u^2 + b_1^2, \\ x_1 - x_j = b_j, \quad j \geq 2. \end{cases} \tag{15}$$

We can restate these equations (renaming $x := x_1$) as:

$$x \prod_{j \geq 2} (x - b_j) - b_1^2 = u^2. \tag{16}$$

Define $P_b(x) = x \prod_{j \geq 2}(x - b_j) - b_1^2$. We can regard $P_b(x)$ as a polynomial in the variable x with $b \in \mathbb{R}^n$ acting as a parameter. We notice that for all values of b, $P_b(x) = 0$ has always a non-negative *real* solution. Define $\mathcal{Z}_b^{\mathbb{R}} = \{\zeta(b) \in \mathbb{R} \mid P_b(\zeta(b)) = 0\}$. This is an ordered set, so we can take $\zeta_0(b) = \max \mathcal{Z}_b^{\mathbb{R}}$. We observe that $P_b(x) > 0$ for $x > \zeta_0(b)$; $P_b'(x) \neq 0$ for $x > \zeta_0(b)$ and $P_b'(\zeta_0(b)) = 0$ if and only if $b \in \Delta$. Observe that $\zeta_0(b)$ becomes a multiple root of $P_b(x)$ when $b \in \Delta$.

Lemma 4.10. *The function $\zeta_0(b)$ is smooth on $\mathbb{R}^n \setminus \Delta$ and continuous on Δ. Let $\partial_{J_k}^k \zeta_0$ denote an order k partial derivative of ζ_0, $J_k = j_1, \ldots, j_n$, $j_1 + \cdots + j_n = k$. Let $B \subset \mathbb{R}^n$ be a small neighbourhood of $0 \in \mathbb{R}^n$. Then,*

$$\partial_{J_k}^k \zeta_0 = \sum_{l < \infty} \frac{G_l(b, x)|_{\zeta_0}}{(P_b'(\zeta_0))^{\lambda_l}}, \tag{17}$$

where $G_l(b, x)$ is bounded on B and $\lambda_l \in \mathbb{Z}^+$ is a finite power.

Proof. For $b \in \mathbb{R}^n \setminus \Delta$, $P_b'(\zeta_0(b)) \neq 0$ and it follows that $\zeta_0(b)$ is smooth on $\mathbb{R}^n \setminus \Delta$. Let $G(b) = P_b(\zeta_0(b))$ and consider $\partial_{b_j} G$. We notice that $G \equiv 0$ on $\mathbb{R}^n$, hence $\partial_{b_j} G \equiv 0$. This implies

$$\partial_{b_j} \zeta_0 = -\frac{\partial_{b_j} P_b\big|_{\zeta_0}}{P_b'(\zeta_0)}.$$

The function $G_{1_j}(b) = \partial_{b_j} P_b\big|_{\zeta_0}$ is bounded on a small ball $B \subset \mathbb{R}^n$ centred at $0 \in \mathbb{R}^n$. The verification of the case $k > 1$ is left to the reader. $\square$

Let $\epsilon > 0$ and let $\zeta_\epsilon(b)$ be the maximal real solution of $P_b(x) - \epsilon^2 = 0$. Observe that $\zeta_0(b) < \zeta_\epsilon(b)$ and $P_b'(\zeta_\epsilon(b)) \neq 0$ for all $b \in \mathbb{R}^n$. It is easy to verify that $\zeta_\epsilon(b)$ is a smooth function on $\mathbb{R}^n$.

Corollary 4.11. *Let $F : \mathbb{C}^n \to \mathbb{R}^3$ as in (11). Let ζ_0 and ζ_1 be the maximal real solutions of $P_b(x) = 0$ and $P_b(x) - 1 = 0$ respectively. Let $\theta_\pm(b) = \mathrm{Arg}(\pm 1 + ib_1)$. The maps Σ^- and Σ^+,*

$$\Sigma^\pm(b) = (\sqrt{\zeta_1(b)} \cdot e^{i\theta_\pm(b)}, \sqrt{\zeta_1(b) - b_2}, \ldots, \sqrt{\zeta_1(b) - b_n}), \tag{18}$$

are sections of F which are smooth on $\mathbb{R}^n$. Let $\theta_0(b) = \mathrm{Arg}(ib_1)$. The section

$$\Sigma^0(b) = (\sqrt{\zeta_0(b)} \cdot e^{i\theta_0(b)}, \sqrt{\zeta_0(b) - b_2}, \ldots, \sqrt{\zeta_0(b) - b_n})$$

is smooth on $\mathbb{R}^n \setminus \Delta$ and continuous on $\mathbb{R}^n$.

Proof. It remains to verify that the above maps are sections of F. Let π as in (14). A direct computation shows that $\pi(\Sigma^\pm(b)) = (\pm 1 + ib_1, b_2, \cdots, b_n)$ and $\pi(\Sigma^0(b)) = (ib_1, b_2, \ldots, b_n)$. Since F factors via π in an obvious way, the claim follows. $\square$

Now let ϕ_i^t be the flow of the Hamiltonian vector field V_{F_i} and consider the Poisson $\mathbb{R}^n$-action, $\Phi : \mathbb{R}^n \times \mathbb{C}^n \to \mathbb{C}^n$:

$$\Phi(t_1, \dots, t_n; z) = \phi_1^{t_1} \circ \cdots \circ \phi_n^{t_n}(z). \tag{19}$$

Remark 4.12. Observe that Φ is free and transitive along the fibres of F over $\mathbb{R}^n \setminus \Delta$. Let $b_0 \in \mathbb{R}^n \setminus \Delta$. Then, for each $z = \Sigma^-(b_0)$ there is $(\alpha_1^0, \dots, \alpha_n^0) \in \mathbb{R}^n$ such that $\Phi(\alpha_1^0, \dots, \alpha_n^0; z) \in \Sigma^+(b_0)$. It follows from similar arguments to the ones used in the proof of Lemma 3.9 that there are locally defined C^∞ functions $\alpha_i(b)$ on $\mathbb{R}^n \setminus \Delta$ such that $\alpha_i(b_0) = \alpha_i^0$ and such that $\Phi(\alpha_1(b), \dots, \alpha_n(b), z) \in \Sigma^+(\mathbb{R}^n \setminus \Delta)$ for all $z \in \Sigma^-(\mathbb{R}^n \setminus \Delta)$.

Denote by $\alpha := \alpha_1$ and ϕ^t the flow of V_{F_1}. Let $\mathcal{O}^-(b)$ and $\mathcal{O}^+(b)$ the T^{n-1}-orbits of $\Sigma^-(b)$ and $\Sigma^+(b)$ respectively; it follows that $\mathcal{O}^-(b) \cong \mathcal{O}^+(b) \cong T^{n-1}$.

It is easy to see that for $x \in \mathcal{O}^-(b)$, $\phi^{\alpha(b)}(x) \in \mathcal{O}^+(b)$. Let $z(b) \in \mathcal{O}^-(b)$ and $w(b) \in \mathcal{O}^+(b)$. Let φ denote the flow of $\pi_*(V_{F_1})$. It is straightforward to check that the solution to the equation $\varphi^{t(b)}(\pi(z(b))) = \pi(w(b))$ is precisely $t(b) = \alpha(b)$. We want to find an explicit expression of $\alpha(b)$. An easy computation shows that $\pi_*(V_{F_1}) = -\chi \partial_u$ where,

$$\chi = \sum_{j=1}^{n} \frac{\prod_{i=1}^n |z_i|^2}{|z_j|^2}. \tag{20}$$

Using formulae (15), we see that $\chi = \partial_x P_b(x)$. Regarding $\mathbb{C} \times \mathbb{R}^{n-1} = \mathbb{R}^{n+1}$ with coordinates (u, b), we can write $\pi_*(V_{F_1})$ as the vector field in $\mathbb{R}^{n+1}$:

$$-2u\partial_x u \frac{\partial}{\partial u} \tag{21}$$

Observe that for $b \in \mathbb{R}^n \setminus \Delta$ this vector field is not singular.

Lemma 4.13. *Let V be a vector field over $\mathbb{R}$. Let p_0 and p be two points in $\mathbb{R}$ and assume $V(u) \neq 0$ for $u \in [p_0, p]$. The time it takes to flow from p_0 to p is:*

$$T = \int_{p_0}^{p} \frac{du}{V(u)}.$$

Proof. Let $\varphi(t, u)$ be the flow of V. We want to find the time $T = T(p)$ such that $\varphi(T, p_0) = p$. We point out that $\partial_t \varphi(t, p_0)|_{t=T(u)} = V(u)$. Then the derivative of $\varphi(T(u), p_0)$ with respect to u is $V(u)\partial_u T(u) = 1$. The claim follows easily from this. ∎

Proposition 4.14. *The function α is hypergeometric. Explicitly,*

$$\alpha(b) = -\int_{\zeta_0(b)}^{\zeta_1(b)} \frac{dx}{\sqrt{P_b(x)}}, \quad b \in \mathbb{R}^n \setminus \Delta, \tag{22}$$

where $\zeta_0(b)$ is the maximal real root of $P_b(x) = x(x-b_2)\cdots(x-b_n) - b_1^2$ and $\zeta_1(b)$ is the maximal real solution of $P_b(x) - 1 = 0$.

Proof. First observe that $\pi(z(b)) = (-1, b)$ and $\pi(w(b)) = (1, b)$. It follows from Lemma 4.13 that:

$$-\alpha(b) = \int_{(-1,b)}^{(1,b)} \frac{du}{2u\partial_x u}. \tag{23}$$

Bearing in mind that $u = \pm\sqrt{P_b(x)}$, it is not difficult to see that α is as claimed. ∎

Of course, we can integrate α explicitly only when $n = 2$. We are particularly interested in the case when $n \geq 3$, for which we need a precise understanding of the behaviour of $\alpha(b)$ as $b \to \Delta$. Let us write,

$$P_b = (x - \zeta_0)Q_b(x),$$

where $Q_b(x)$ is a polynomial with real coefficients. We notice that $\zeta_0(b)$ becomes a (possible multiple) root of $Q_b(x)$ if and only if $b \in \Delta$.

Proposition 4.15. *Let α as above and let $\partial^k_{J_k}\alpha$ denote a partial derivative of order k. Then, α is bounded from above by*

$$-\frac{2}{(Q_b(\zeta_0))^{\frac{1}{2}}}. \tag{24}$$

There are finite powers $w_0, w_1, \ldots, w_{n-1} \in \mathbb{Z}^+$, depending on J_k, such that near Δ,

$$|\partial^k_{J_k}\alpha| \simeq \frac{1}{P_b'(\zeta_0)^{w_0}|\zeta_0 - \beta_1|^{w_1} \cdots |\zeta_0 - \beta_{n-1}|^{w_{n-1}}}, \tag{25}$$

where $\beta_i(b) = \operatorname{Re}\rho_i(b)$ are the real part of the roots of Q_b, $P_b = (x - \zeta_0)Q_b(x)$.

Proof. The proof involves the use of a (truncated) asymptotic expansion of α. Since the integration limits of α depend on b, the estimates of $\partial^k_{J_k}\alpha$ turn out to be rather messy, as they involve the derivatives of ζ_0. Here we estimate α to order $k = 0$ and refer the reader to [1] for the details concerning $k \geq 1$.

Let $I = \alpha$ and let $f = Q_b^{-\frac{1}{2}}$ and $dg = (x - \zeta_0)^{-\frac{1}{2}}dx$. Integrating $I = \int f dg$ by parts we get,

$$I = 2\frac{(x-\zeta_0)^{\frac{1}{2}}}{Q_b(x)^{\frac{1}{2}}}\Bigg|_{\zeta_0}^{\zeta_1} - \int_{\zeta_0}^{\zeta_1} -\frac{(x-\zeta_0)^{\frac{1}{2}}Q_b'(x)dx}{(Q_b(x))^{1+\frac{1}{2}}}. \tag{26}$$

Let R_1 be the first summand on (26), and let I_1 be the integral. We notice that $R_1 = 2(Q_b(\zeta_1))^{-\frac{1}{2}}$. Since x is such that $0 \leq x - \zeta_0(b) \leq 1$, then

$$0 \geq I_1 \geq \int_{\zeta_0}^{\zeta_1} -\frac{Q_b'(x)dx}{(Q_b(x))^{1+\frac{1}{2}}} = 2[(Q_b(\zeta_1))^{-\frac{1}{2}} - (Q_b(\zeta_0))^{-\frac{1}{2}}] \tag{27}$$

Then we get, $|I| \simeq 2(Q_b(\zeta_0))^{-\frac{1}{2}}$. □

Remark 4.16. What Proposition 4.15 says is that the derivatives of α blow up at Δ when the ζ_0 becomes a multiple root of $Q_b(x)$. Furthermore, α and all its derivatives are bounded by a rational function having a pole of certain finite order along Δ. For instance, when $n = 3$, ζ_0 becomes a root of $Q_b(x)$ as b approaches to the spokes of Δ, so α blows up there with order at most $-\frac{1}{2}$. This root becomes double at $0 \in \mathbb{R}^3$, so α blows up there with order at most -1.

Now let $B \subseteq \mathbb{R}^n$ be an open neighbourhood of $0 \in \mathbb{R}^n$, let $B_0 := B \setminus \Delta$ and let $\alpha_1, \ldots, \alpha_n$ as in Remark 4.12. Define a map $A : \Sigma^-(B_0) \to \Sigma^+(B_0)$ as $A(z) = \Phi(\alpha_1, \ldots, \alpha_n; z)$. In view of (18), we can write A explicitly as $A : (z_1, z_2, \ldots, z_n) \mapsto (-\bar{z}_1, z_2, \ldots z_n)$. We verify that A is smooth, furthermore, A extends smoothly to $z \in \Sigma^-(B)$, regardless of the fact that the Poisson action is not freely transitive over singular fibres.

Let $\tau_0 = \sum \alpha_j db_j$. We can find a 1-form $\eta = \sum \eta_j db_j$ on B such that $\tau := \tau_0 + \eta$ is closed. Indeed, let σ be such that $d\sigma = \omega$ and let $\gamma(b)$ be a curve joining $\Sigma^-(b)$, and $\Sigma^+(b)$. One can verify that $\tau_0 = dA_\gamma + R_\gamma$ where $A_\gamma = \int_\gamma \sigma$ and R_γ is a 1-form (c.f. (5)). Defining $\eta = dH - R_\gamma$ for any $H \in C^\infty(B)$, we obtain $\tau = d(A_\gamma + H)$.

Let $A' : \Sigma^+(B) \to \Sigma(B) := A'(\Sigma^+(B))$ be the map, $A'(z) = \Phi(\eta_1, \ldots, \eta_n; z)$. The composition $Q = A' \circ A$,

$$Q : \Sigma^-(B) \to \Sigma(B). \tag{28}$$

is a C^∞ map.

Proposition 4.17. *Let $\tau = \tau_0 + \eta$ be the 1-form as in the paragraph above. Then, there is a symplectic manifold (X, ω) and a Lagrangian fibration $f : X \to B$ such that $\tau_1 := \tau$, and $\tau_j = \pi db_j$, $j = 1, \ldots n$, are the period 1-forms of f. Furthermore, when $n = 3$, f coincides topologically with the example in Construction 4.6.*

Proof. We saw in Proposition 4.15 that the function $\alpha_1(b)$ is bounded from above by $-2(Q_b(\zeta_0(b)))^{-\frac{1}{2}}$. We can find a smaller neighbourhood $B' \subseteq B$ of Δ such that $\alpha_1(b) + \eta_1(b) < 0$ for $b \in B'$. Let $B_0' = B' \setminus \Delta$. Now let $\mathcal{O}_b$ be the subset of $F^{-1}(b)$ defined by:

$$\mathcal{O}_b := \{\Phi(t; z) \mid t \in [\alpha_1(b) + \eta_1(b), 0] \times [0, 2\pi] \cdots \times [0, 2\pi] \subseteq \mathbb{R}^n\}.$$

Let $\overline{U} = \overline{\bigcup_{b\in B'} \mathcal{O}_b}$, this is a T^{n-1}-invariant subset of $\mathbb{C}^n$. We see that for $b \in B'_0$, $\overline{U} \cap F^{-1}(c)$ is a bounded cylinder and for $b \in \Delta$ it is a bounded set in $F^{-1}(b)$. In both cases, the boundary of these sets are the T^{n-1}-orbits: $\mathcal{T}^-(b) = T \cdot \Sigma^-(b)$ and $\mathcal{T}(b) = T \cdot \Sigma(b)$.

Now let $W \subset F^{-1}(B')$ be a small T^{n-1}-invariant neighbourhood of $\Sigma^-(B')$ such that $W \cap Crit(F) = \emptyset$. Let $x \in W$ such that $F(x) = b$. There is a finite $t \in \mathbb{R}^n$, $t = t(x)$, such that $x = \Phi(t, \Sigma^-(b))$. Let $Q : \Sigma^-(B') \to \Sigma(B')$ as in (28). Define a map $\mathcal{Q} : W \to F^{-1}(B')$, $\mathcal{Q}(x) = y = \Phi(t(x), \Sigma(b))$. It follows that $\mathcal{Q}$ extends Q. Moreover, similar arguments to the ones used in Lemma 3.9 can be used to show that $t(x)$ is C^∞. Let $W' := \mathcal{Q}(W)$. Then, $\mathcal{Q} : W \to W'$ is clearly invertible, moreover, $\mathcal{Q}$ is a diffeomorphism and $\mathcal{Q}$ sends $\mathcal{T}^-$ diffeomorphically to $\mathcal{T}$.

Now let $\mathcal{U} = W \cup \overline{U} \cup W'$ and let $x, y \in \mathcal{U}$. We define $X = \mathcal{U}/\sim$ where $x \sim y \Leftrightarrow$ either $x = y$ or $y = \mathcal{Q}(x)$ if $x \in W$ and $y \in W'$. By means of this identification, X is a smooth manifold. Intuitively, $\mathcal{Q}$ identifies the two components on the boundary of $\bar{U}$. F induces a smooth map $f : X \to B'$ such that $f|_V = F|_V$ on a neighbourhood $V \subset int\, \bar{U}$ of $Crit(f)$. If we can make X into a symplectic manifold, then the periods of X are, by construction, $\tau_1, \ldots, \tau_n$.

Now let ω be the standard symplectic structure on $\mathbb{C}^n$. Then ω restricts to symplectic forms on W and W'. Let us consider 1-form $\tau := \tau_1$ which may be multi-valued on B'_0. We can choose a domain $D \subset B'_0$ where τ is single valued. Let V_τ the vector field determined by the equation $F^*\tau = \iota(V_\tau)\omega$. Since τ is closed, V_τ is a symplectic vector field, i.e. its flow, ψ_s, defines a 1-parameter family of symplectomorphisms. In particular, its time $s = 1$ flow map, ψ_1, is a symplectomorphism. One can easily check that $\psi_1|_{W\cap F^{-1}(D)} = \mathcal{Q}|_{W\cap F^{-1}(D)}$. This implies that $\mathcal{Q}$ extends ψ_1 to W, in particular, $\mathcal{Q}^*\omega$ and ω coincide on $W \cap F^{-1}(D)$. It follows that $\mathcal{Q}^*\omega$ and ω coincide on $W \setminus \{f^{-1}(\Delta) \cap W\} \subset W$ which is dense in W. Then, $\mathcal{Q} : W \to W'$ is a symplectomorphism and therefore X is a symplectic manifold. In dimension $n = 3$, it is not hard to see that f coincides topologically with the example given in Construction 4.6. ∎

Remark 4.18. Observe that the symplectic structure in the example in Proposition 4.17 can be deformed by considering the 1-form $\eta' = \eta + dH$ for any C^∞ function H. It turns out that there are cases for which H does not induce a trivial deformation of the symplectic structure. In fact, the example in Proposition 4.17 belongs to a large family of Lagrangian fibrations whose members coincide topologically but may not be symplectomorphic.

Theorem 4.19. *Let $\mathcal{F} = (X, \omega, f) \in \mathcal{L}(1,2)$. Then there are local sections e_1, e_2, e_3 of $R^2_c f^\#_* \mathbb{Z}$ such that the corresponding period 1-forms are:*

$$\tau_1 = \tau_0 + dH, \quad \tau_2 = 2\pi db_2, \quad \tau_3 = 2\pi db_3$$

where $\tau_0 = \sum \alpha_i db_i$ is as in Proposition 4.17, α_1 is as in (22) and H is a smooth function. Let $B \subseteq \mathbb{R}^3$ be an open ball. Secondly, for each $H \in C^\infty(B)$ there is a fibration $\mathcal{F}_H \in \mathcal{L}(1,2)$ with periods τ_1, τ_2, τ_3 as above. The monodromy representation of $\mathcal{F} \in \mathcal{L}(1,2)$ is generated by the matrices: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Proof. The second statement follows from Proposition 4.17 and Remark 4.18. For the first claim, recall from Corollary 4.9 that any $\mathcal{F} \in \mathcal{L}(1,2)$ can be normalised in a neighbourhood $U \subset X$ of $p \in f^{-1}(0) \cap Crit(f)$ by $F : U \cong \mathbb{C}^3 \to B \subset \mathbb{R}^3$, $F = (F_1, F_2, F_3)$ as in (11). By redefining $X := f^{-1}(f(U))$ if necessary, the restriction $f|_{X\setminus U}$ induces a trivial bundle over B with fibre $T^2 \times [0,1]$. We can define sections $e_1, e_2, e_3 \in R^2_c f^\#_* \mathbb{Z}$ in terms of the action of the Hamiltonian vector fields $v_i = v_{F_i}$ on U and their extension to $X \setminus U$. For $i = 2, 3$ and $b \in B$ we take $e_i(b)$ represented by integral curves $\gamma_i : [0, 2\pi] \to F^{-1}(b)$ of v_i. For $e_1(b)$, $b \in B \setminus \Delta$, we consider the sections $\Sigma_1 := \Sigma^+$ and $\Sigma_2 := \Sigma^-$ of F as in Corollary 4.11 and define a representative $\gamma_1(b)$ of $e_1(b)$ as a suitable composition on flows of v_1, v_2, v_3, starting on $\Sigma_1(b)$ passing through $\Sigma_2(b)$ and returning to $\Sigma_1(b)$ in a completely analogous way as we did in the proof of Proposition 3.10. The reader may easily check that the period 1-forms computed over γ_i are τ_i as claimed. ∎

It is well known that the monodromy about the singular fibre of a focus-focus fibration can be explained in terms of a Dehn twist. Similarly, for a fibration $\mathcal{F} \in \mathcal{L}(1,2)$, the monodromy of $\mathcal{F}$ can be understood as a "two dimensional Dehn twist". For each generator of $\pi_1(B \setminus \Delta, b)$, this twist is given by a full turn of a T^2-orbit, $\mathcal{T}(b)$, in one of the following ways:

1. once in the direction of v_2;

2. once in the direction of v_3;
3. the turn in 1) followed by the turn in 2).

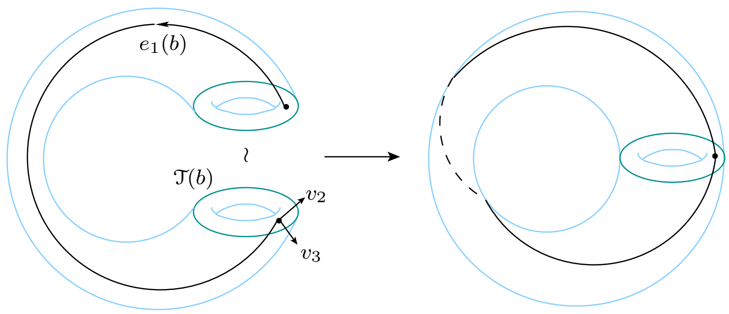

Figure 4: Monodromy around a component of $\Delta - \{0\}$.

In higher dimensions the description is analogous.

5 The classification

Let $\mathcal{F} = (X, \omega, f)$ be a Lagrangian T^3 fibration over a smooth manifold B and let $\Delta \subset B$ be the discriminant locus of f. We are shall consider $\mathcal{F} \in \mathcal{L}(\kappa) := \mathcal{L}(\kappa, 2)$, $\kappa = 1, 2$.

- Case $\kappa = 1$: B is an open 3-ball centred at $b_0 \in \mathbb{R}^3$, Δ is a cone over 3 points. Let $b_0 \in B$ be the vertex of Δ. There is only one singular point p on the fibre X_{b_0}, i.e. the Poisson orbit $O_p = Crit(X_{b_0}) = p$. There is a neighbourhood $U \subset X$ of p and a normal form $\varphi \circ f|_U = q \circ \psi = F$ as in Corollary 4.9. The period lattice of $\mathcal{F}$ is as in Theorem 4.19.
- Case $\kappa = 2$: $B = D \times (0, 1)$, $\Delta = \{0\} \times (0, 1)$. Let $b_0 \in \Delta$ and X_{b_0} the fibre over b_0. A point $p \in Crit(f)$ over b_0 belongs to a Poisson orbit $O_p = Crit(X_{b_0}) \cong S^1$. There is a neighbourhood $U \subset X$ of O_p and a normal form $\varphi \circ f|_U = q \circ \psi = F$ as in Definition 3.6. The period lattice of $\mathcal{F}$ is as in Proposition 3.10.

Definition 5.1. Let (b_1, b_2, b_3) be coordinates on B and let $\phi \in C^\infty(B)$. Let $\partial_{J_k}\phi$ denote a partial derivative of ϕ. We say that ϕ is *k-flat* at Δ if $\partial_{J_k}\phi|_b = 0$ for each $b \in \Delta$ and each $J_k \leq k$. If $k = \infty$ we say that ϕ is *flat* at Δ. Let $\varphi : B \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ map, written as $\varphi = (\varphi_1, \varphi_2, \varphi_3)$. We say that φ is tangent to the identity at Δ if for each $i = 1, 2, 3$, the function $\varphi_i(b) - b_i$ is flat at Δ.

Definition 5.2. Let α be a function on B which is C^∞ on $B \setminus \Delta$. We say that α is of *rational type* if for each $J_k \in \mathbb{Z}_{\geq 0}$ and any flat function ϕ,

$$\lim_{b \to \Delta} \phi \partial_{J_k} \alpha = 0.$$

Example 5.3. The function $\alpha(s, r) = \log|s|$ on $D \times (0, 1)$ is of rational type. Similarly, α as in Proposition 4.14 is also of rational type.

Definition 5.4. Let $\mathcal{F}, \mathcal{F}' \in \mathcal{L}(\kappa)$, $\kappa = 1, 2$, and let $\tau = \tau_0 + dH$ and $\tau' = \tau_0 + dH'$ be the singular periods of $\mathcal{F}$ and $\mathcal{F}'$ respectively. We say that $\mathcal{F}$ is *formally equivalent* to $\mathcal{F}'$ if the function $H - H'$ is flat at Δ.

Proposition 5.5. *Let $\mathcal{F} = (X, \omega, f)$ and $\mathcal{F}' = (X', \omega', f')$ in $\mathcal{L}(\kappa)$. Let τ_i and τ'_i, $i = 1, 2, 3$, be the corresponding period 1-forms. If $\mathcal{F}$ and $\mathcal{F}'$ are formally equivalent there is a C^∞-diffeomorphism between two small neighbourhoods of $b_0 \in \Delta$, $\varphi : B \to \varphi(B) =: B'$, such that $\varphi^*(\tau'_i) = \tau_i$ for $i = 1, 2, 3$. Conversely, if there is a diffeomorphism φ of B matching τ_i and τ'_i and such that φ is tangent to the identity, then $\mathcal{F}$ is formally equivalent to $(X', \omega', \varphi^{-1} \circ f')$.*

Proof. Let $\tau := \tau_1$ and $\tau' := \tau'_1$ be the singular periods of $\mathcal{F}$ and $\mathcal{F}'$, expressed in the coordinates (b_1, b_2, b_3) as in the previous section. We want to find a diffeomorphism φ such that $\varphi^*\tau' = \tau$ and $\tau_j = \varphi^*\tau'_j$, $j = 2, 3$. The latter implies that φ should be of the form:

$$b \mapsto (\varphi_1(b), b_2, b_3), \tag{29}$$

where φ_1 is a smooth function to be determined. Now, for $t \in [0,1]$, we define a family of closed 1-forms $\tau_t = \tau + t(\tau' - \tau)$. Then, $\tau_t|_{t=0} = \tau$ and $\tau_t|_{t=1} = \tau'$. Suppose there is a 1-parameter family of maps, G_t, varying smoothly with respect to t, such that each G_t is a diffeomorphism between small neighbourhoods of $0 \in \mathbb{C}$ and such that G_0 is the identity map. Additionally, suppose that:

$$\frac{dG_t^*\tau_t}{dt} = 0. \tag{30}$$

Then, $G_1^*\tau_{t_1} = G_0^*\tau_{t_0}$ and we could define $\varphi := G_1$. It is standard to realise G_t by means of integrating a time dependent vector field V_t. Using Cartan identity, we can rewrite equation (30) as:

$$G_t^*\left(\mathcal{L}_{V_t}\tau_t + \frac{d\tau_t}{dt}\right) = G_t^*\left(d(\iota_{V_t}\tau_t) + \tau' - \tau\right) = 0. \tag{31}$$

Observe that $\tau' - \tau = d(H' - H)$. Then, the solution to (31) is determined by

$$\iota_{V_t}\tau_t = H - H'. \tag{32}$$

The solution should be of the form $V_t = g_t(b)\partial_{b_1}$ where $g_t(b)$ is a smooth function of b and t. We observe that, since we want $\varphi(\Delta) = \Delta$, then V_t should satisfy $V_t(\Delta) = 0$. The left hand side of equation (32) is:

$$g_t(b) \cdot \Big(\alpha(b) + \frac{\partial}{\partial b_1}\big(H + t(H' - H)\big)\Big). \tag{33}$$

Define

$$g_t(b) = \frac{H - H'}{\alpha(b) + \psi(b,t)}$$

where $\psi(b,t) = \frac{\partial H}{\partial b_1} + t\frac{\partial(H'-H)}{\partial b_1}$ is a C^∞ function on $B \times [0,1]$. For $\kappa = 1$ we know from Remark 4.16 that α blows up at Δ with order at most -1. In the other hand, for $\kappa = 2$ α blows up at Δ as a logarithm. Since $H - H'$ vanishes at Δ to all orders, for $\kappa = 1, 2$ we have

$$\lim_{b \to \Delta} \frac{(H - H')}{\alpha} = 0.$$

Therefore g_t is continuous and $g_t(b) = 0$ when $b \in \Delta$. In particular $V_t(\Delta) = 0$ as required. A similar argument can be used to prove the smoothness of g_t. Indeed, for $\kappa = 1$ the estimates carried out in Proposition 4.15 show that all the functions $\partial_{J_k}\alpha$ blow up with finite order along Δ. This implies that for any $h \in C^\infty(B)$ which is flat on Δ,

$$\lim_{b \to \Delta} h\partial_{J_k}\alpha = 0.$$

Now observe that the k-th partial derivatives of $(H - H')/\alpha$ are finite sums of terms of the type

$$\frac{h\partial_{J_k}\alpha}{\alpha^m}$$

with $m \leq 2k$ and $h \in C^\infty(B)$ flat at Δ. It is not difficult to see from this that $\partial_{J_k} g_t$ is a continuous function on B which vanishes at Δ. A completely analogous argument is valid for the case $\kappa = 2$. Therefore $g_t(s)$ is a C^∞ function on B and it is flat at Δ. This implies that the time one map of $V_t(b)$, $\varphi := G_1$, is a diffeomorphism which is tangent to the identity on Δ.

Now suppose there is a diffeomorphism φ matching τ_i and τ_i' which is tangent to the identity at Δ. Since $\varphi^*\tau_1' = \tau_1$ we can write $\varphi^*\tau_0 - \tau_0 = d(H - H' \circ \varphi)$. Furthermore, φ can be written in coordinates (b_1, b_2, b_3) as before as $\varphi = (\varphi_1, b_2, b_3)$, with $\varphi_1 = \varphi_1(b)$ a smooth function on B. Observe that the 1-form $T := \varphi^*\tau_0 - \tau_0$ is single valued and smooth on B. Let us write $T = \sum T_i db_i$. We claim that the functions T_i are flat on Δ. From this it follows directly that $H - H' \circ \varphi$ is flat at Δ. Now we will see that $\partial_{J_k} T_i|_\Delta = 0$ for all $i = 1, 2, 3$ and for each $J_k \leq k$, $k \in \mathbb{Z}_{\geq 0}$. Recall that $\tau_0 = \sum \alpha_j db_j$, where $\alpha_1 = \alpha$ is a function on B of rational type (cf. Definition 5.2) and α_2 and α_3 are locally defined. After an easy calculation we obtain

$$\begin{aligned} T_1 &= (\alpha_1 \circ \varphi)\partial_{b_1}\varphi_1 - \alpha_1 \\ T_2 &= (\alpha_1 \circ \varphi)\partial_{b_2}\varphi_1 + \alpha_2 \circ \varphi - \alpha_2 \\ T_3 &= (\alpha_1 \circ \varphi)\partial_{b_3}\varphi_1 + \alpha_3 \circ \varphi - \alpha_3. \end{aligned} \tag{34}$$

Since φ is tangent to the identity at Δ, $\partial_{b_1}\varphi_1|_\Delta = 1$ and $\partial_{b_1}\varphi_1(b) > 0$ for b in a small enough neighbourhood of Δ. Then $T_1(b) \to 0$ as $b \to \Delta$ but since T_1 is continuous we have $T_1|_\Delta = 0$. Similarly, $\partial_{b_2}\varphi_1$, $\partial_{b_3}\varphi_1$ and all the higher order derivatives of φ_1 vanish when restricted to Δ. In particular $\partial_{b_2}\varphi_1$, $\partial_{b_3}\varphi_1$ are flat on Δ. Then $(\alpha_1 \circ \varphi)\partial_{b_j}\varphi_1|_\Delta = 0$ for $j = 2, 3$. Now let us consider a (perhaps smaller) neighbourhood of Δ and a branch of α_j in this neighbourhood. Since φ is infinitely tangent to the identity at Δ, then $(\alpha_j \circ \varphi - \alpha_j)(b) \to 0$ as $b \to \Delta$. From the continuity of T_j we have $T_j|_\Delta = 0$ for $j = 2, 3$. One can use the argument above inductively to show that $\partial_{J_k} T_i$ vanish on Δ for $k \geq 1$. □

Proposition 5.6. *Let $\mathcal{F}$ and $\mathcal{F}'$ be in $\mathcal{L}(\kappa)$. Let $\varphi : B \to \varphi(B) =: B'$ be a diffeomorphism, such that $\varphi(\Delta) = \Delta$ and $\varphi^*(\tau_i') = \tau_i$ for $i = 1, 2, 3$. Then, there are sections Σ and Σ' of f and f' and a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ B \arrow[r, "\varphi"] & B' \end{tikzcd}$$

where Φ is an orientation preserving diffeomorphism sending Σ to Σ'. The map Φ can be assumed to be equivariant with respect to the T^2-actions induced by τ_j and τ_j' $j = 2, 3$. Furthermore, if Σ and Σ' are Lagrangian, then Φ is a symplectomorphism.

Proof. Recall that $\mathcal{F}$ is normalised in a neighbourhood $U \subset X$ of the critical Poisson orbit $O_p \subset X_{b_0}$ by means of a symplectomorphism $\psi : U \to (V, \omega_0)$. Similarly, for $\mathcal{F}'$ there is a neighbourhood $U' \subset X'$ of $O_{p'}$ and a symplectomorphism $\psi' : U' \to (V, \omega_0)$.

Let $W \subseteq \psi(U) \cap \psi'(U')$. For simplicity denote $U := \psi^{-1}(W)$ and define $\Phi_0 := (\psi')^{-1} \circ \psi|_U$. Then, Φ_0 is a symplectomorphism such that $\Phi_0(O_p) = O_{p'}$ and such that $\varphi \circ F = F' \circ \Phi_0$. Now let Σ be a section of F which does not pass through O_p. Defining $\Sigma' = \Phi_0(\Sigma)$ gives a section of F' which does not pass through $O_{p'}$. Notice that Σ and Σ' also define sections of f and f' respectively. Since Φ_0 is a symplectomorphism, if Σ is Lagrangian, then Σ' is Lagrangian too.

Let α be a local section of T_B^*. Let v_α be the vector field determined by the equation:

$$F^*\alpha = \iota(v_\alpha)\omega. \tag{35}$$

If we consider the 1-form, db_i, then $v_{db_i} = v_{q_i}$. As we observed before, each v_{q_i} extends to a vector field on X which is tangent to the fibres of f. Therefore, v_α extends to X and, since the fibres of f are compact, the flow g_α^t of v_α is defined for all $t \in \mathbb{R}$. For each α define the map $T_\alpha := g_\alpha^1 : X \to X$. It follows that $\alpha \mapsto T_\alpha$ induces a fibre preserving action, $T : T_B^* \times_B X \to X$. Now define the map $\tilde{\pi} : T_B^* \to X$ such that for each $\alpha_b \in T_{B,b}^*$, $\tilde{\pi}(\alpha_b) = T_\alpha(\Sigma(b)) =: x$, which lies on the fibre $f^{-1}(b)$. One can verify that x only depends on the value of α at b. So, for $\bar{\alpha}$ such that $\bar{\alpha}(b) = \alpha(b)$, $T_{\bar{\alpha}}(\Sigma(b)) = T_\alpha(\Sigma(b)) = x$.

Let Z be the zero section on T_B^*. We know from Theorem 2.5 that $\tilde{\pi}(T_B^*) = X^\#$, $\tilde{\pi}^{-1}(\tilde{\pi}(Z)) = \Lambda$ is the period lattice of f and $\tilde{\pi}^{-1}|_{X^\#} : X^\# \to T_B^*$ is well defined modulo Λ. Moreover, $\tilde{\pi}^{-1}|_{X^\#}$ composed with the projection $T_B^* \to T_B^*/\Lambda = J_f$ gives a diffeomorphism $X^\# \cong J_f$. If Σ is Lagrangian this map is a symplectomorphism.

Now let us take $\alpha' = (\varphi^{-1})^*\alpha$; this is a local section of $T_{B'}^*$. Consider the vector field $v_{\alpha'}$ induced by $(F')^*\alpha' = \iota(v_{\alpha'})\omega'$. Let $g_{\alpha'}^t$ be the flow of $v_{\alpha'}$. Again, this flow is complete, so we can define $T_{\alpha'} : X' \to X'$ such that $T_{\alpha'} := g_{\alpha'}^1$. Let $\tilde{\pi}' : T_{B'}^* \to X'$ such that $\tilde{\pi}'(\alpha_{b'}') = T_{\alpha'}(\Sigma'(b')) =:$ x'. Define the map $\Phi^\# : X^\# \to X'^\#$ as the composition:

$$\begin{tikzcd} x \in X^\# \arrow[r, "\Phi^\#"] \arrow[d] & x' \in X'^\# \\ {[\alpha_b] \in J_f} \arrow[r, "(\varphi^{-1})^*"'] & {[\alpha_b'] \in J_{f'}} \arrow[u] \end{tikzcd} \tag{36}$$

The horizontal map, which is induced by the pull back of sections under φ, is well defined as φ^* sends Λ' to Λ. The vertical maps, e.g. $[\alpha_b] \mapsto g_\alpha^1(\Sigma(b))$, are independent of the choice of the representative of $[\alpha_b] \in J_f$. Indeed, let $\tilde{\alpha}_b \in [\alpha_b]$. Then, $\tilde{\alpha}_b = \alpha_b + \lambda_b$, with $\lambda_b \in \Lambda_b$. It follows that $g_{\alpha+\lambda}^t = g_\alpha^t$. In particular, $g_{\alpha+\lambda}^1(\Sigma(b)) = g_\alpha^1(\Sigma(b))$.

We can write explicitly,

$$\Phi^{\#}(x) = g^1_{\alpha'}(\Sigma'(b')) \tag{37}$$

where $x = g^1_\alpha(\Sigma(b))$ for some $[\alpha_b] \in J_f$ and $\alpha' = (\varphi^{-1})^*\alpha$. Notice that φ induces a symplectomorphism between $T^*_{B'}$ and T^*_B. Hence $\Phi^{\#}$ is a diffeomorphism and, when Σ is Lagrangian, $\Phi^{\#}$ is a symplectomorphism.

Now let $X^{\#} \hookrightarrow X$ be the inclusion map and consider $x \in U \cap X^{\#}$ over $b \in B$. We define

$$\Phi(x) = \begin{cases} \Phi^{\#}(x), & x \in X^{\#}, \\ \Phi_0(x) & x \in U. \end{cases}$$

The map Φ extends $\Phi^{\#}$ to X and the T^2-equivariance of Φ is verified *a priori*. Φ is C^∞ since the map $J_f \to J_{f'}$ is. We still need to check, however, that $\Phi^{\#}(x) = \Phi_0(x)$ on $U \cap X^{\#}$, i.e. that Φ is well defined. We prove this next.

Let $x \in U \cap X^{\#}$ over $b \in B$ and define $v'_\alpha := \Phi_{0*}(v_\alpha)$ and let $g^t_{v'_\alpha}$ denote the flow of v'_α. We claim that the equation $x = g^1_\alpha(\Sigma(b))$ implies that

$$\Phi_0(x) = g^1_{v'_\alpha}(\Phi_0(\Sigma(b))) \tag{38}$$

To see this let us regard $\gamma(t) := g^t_\alpha(\Sigma(b))$ as the integral curve of v_α such that $\gamma(0) = \Sigma(b)$ and $\gamma(1) = x$. Now let $\gamma'(t) := \Phi_0(\gamma(t))$. This is a curve on $F'^{-1}(b')$, $b' = \varphi(b)$, such that $\gamma'(0) = \Phi_0(\Sigma(b)) = \Sigma'(b')$ and $\gamma'(1) = \Phi_0(x)$. Furthermore, γ' is an integral curve of v'_α. Indeed, we see that:

$$\frac{d\gamma'}{dt} = \frac{d(\Phi_0 \circ \gamma)}{dt} = \Phi_{0*}(v_\alpha) = v'_\alpha.$$

Therefore $\gamma'(t) = g^t_{v'_\alpha}$ and $g^1_{v'_\alpha}(\Sigma'(b')) = \Phi_0(x)$. Now observe that $\Phi_0^*\omega' = \omega$ implies that:

$$v'_\alpha = v_{\alpha'}. \tag{39}$$

To prove this we notice that $F^*\alpha = \Phi_0^*(F'^*\alpha')$. Now we can write (35) as:

$$\Phi_0^{-1*}(\iota(v_\alpha)\Phi_0^*\omega') = F'^*\alpha' \tag{40}$$

The left hand side of (40) can be written as $\iota(\Phi_{0*}v_\alpha)\omega'$. Then, it follows that $v'_\alpha = \Phi_{0*}v_\alpha = v_{\alpha'}$. Now, from (38) and (39) we conclude that:

$$\Phi_0(x) = g^1_{\alpha'}(\Sigma'(b'))$$

which is equal to $\Phi^{\#}(x)$ in (37), hence Φ is well defined.

Observe that, for $\kappa = 2$, we can start the above construction in terms of the section $\Sigma = \Sigma_1$ as in Construction 3.8, which is Lagrangian. Therefore Φ turns out to be a symplectomorphism. In the case $\kappa = 1$, the sections $\Sigma^\pm$ as in (18) are not Lagrangian. This does not give much trouble as we can always find a Lagrangian section. The argument is valid for $\kappa = 1, 2$. Let $U \subset X$ be as before. Observe that for any given section Σ_0 of f with $\Sigma_0(B) \subset U$, there exists a neighbourhood $\mathcal{U} \subseteq U \subset X$ of Σ_0 such that $\mathcal{U} \cap Crit(f) = \varnothing$ and a fibre-preserving symplectomorphism $(\mathcal{U}, \omega|_{\mathcal{U}}) \to (T^*_B, \Omega)$. Here $\Omega = \omega_0 + \beta$ where ω_0 is the standard symplectic structure on T^*_B and β is the pull-back under $T^*_B \to B$ of a closed 2-form on B (if Σ_0 were Lagrangian $\beta = 0$). Observe that in our situation β can be assumed to be exact, so we have $d\theta = \Omega - \omega_0$ for some 1-form θ on B. Then $-\theta$ defines a section, Σ_θ, of T^*_B which is Lagrangian with respect to Ω. Then Σ_θ maps to a Lagrangian section, Σ, of f inside $\mathcal{U}$. Using Σ to define Φ we obtain a symplectomorphism. $\square$

Theorem 5.7. *Let $\mathcal{F} = (X, \omega, f)$ and $\mathcal{F}' = (X', \omega', f')$ be Lagrangian fibrations of type $\mathcal{L}(\kappa)$, $\kappa = 1, 2$. Then $\mathcal{F}$ is formally equivalent to $\mathcal{F}'$ if and only if $\mathcal{F}$ is symplectically equivalent to $\mathcal{F}'$.*

Proof. Assume $\mathcal{F}$ and $\mathcal{F}'$ are formally equivalent. Then Proposition 5.5 gives us a diffeomorphism φ on B such that $\varphi^*\tau'_j = \tau_j$. In view of Proposition 5.6, φ lifts to a fibre-preserving symplectomorphism $\Phi : X \to X'$.

To prove the converse we suppose there is a symplectomorphism Ψ and a suitable diffeomorphism φ, making a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\Psi"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ B \arrow[r, "\varphi"] & B' = \varphi(B) \end{tikzcd} \tag{41}$$

One can always take a diffeomorphism, $\tilde{\varphi}$, from a neighbourhood of $b_0 \in B$ into a neighbourhood, $\tilde{B} \subseteq B'$, of $\varphi(b_0) = b_0'$ and such that $\varphi \circ \tilde{\varphi}^{-1}$ is tangent to the identity at $\Delta \cap \tilde{B}$. Let $\tilde{f} = \tilde{\varphi} \circ f$. Then, $(X, \omega, \tilde{f})$ and $\mathcal{F}$ define the same germ. Now, Ψ together with the map $\varphi' := \varphi \circ \tilde{\varphi}^{-1} : \tilde{B} \to \varphi'(\tilde{B}) \subseteq B'$ makes $(X, \omega, \tilde{f})$ and $\mathcal{F}'$ symplectomorphic, with φ' being tangent to the identity at Δ. Let us denote $f := \tilde{f}$ and $\varphi := \varphi'$.

We claim now that $\tau_i = \varphi^* \tau_i'$. To see this we take V_i' to be the vector fields determined by the equation

$$f'^* \tau_i' = \iota_{V_i'} \omega', \qquad i = 1, 2, 3. \tag{42}$$

These vector fields are defined on open sets $f'^{-1}(U')$, where $U' \subset B_0'$ is an open set on which a branch of τ_1' is defined. It follows that V_i' are vector fields whose flows are periodic. We can take integral curves of V_i' to define simple loops, $\gamma_i'(b)$ and on $f'^{-1}(b)$, representing the cycles $e_i'(b)$ generating $H_1(f'^{-1}(b'), \mathbb{R})$. These loops can be used for computing the period 1-forms of f' which are, tautologically, τ_i'. Now define V_i to be the vector fields determined by the equation $\iota(V_i)\omega = (f' \circ \Psi)^* \tau_i$. Since Ψ is symplectic, $\Psi_* V_i = V_i'$. The above implies that the flow of V_i is periodic. One verifies that suitable integral curves γ_i of V_i generate $H_1(X_b, \mathbb{Z})$, so we can define the period one forms, τ_i of f by integrating along γ_i. Now observe that, since the diagram (41) commutes, V_i also satisfies the equation $\iota(V_i)\omega = \varphi^* \tau_i'$. Therefore, $\tau_i = \varphi^* \tau_i'$. The conclusion follows now from Proposition 5.5. $\square$

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